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A Proof of Hales-Jewett Theorem using a Non-Standard Method

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Shelah's proof of Hales-Jewett theorem was explained in [1]. Following the ideas explained there, I give a proof using a nonstandard method.

Theorem 1. *For all $n, n_c \in \omega$, we can find $N = N_{n, n_c} \in \omega$ with the following property: For a given coloring $c : n^N \rightarrow n_c$ there is a word $w(x) \in (n \cup \{x\})^N \setminus n^N$ such that $\{w(i) : i \in n\}$ is c -monochromatic.*

Proof. Throughout n_c is fixed. We prove the theorem by induction on n . So we assume that the theorem is true for n . Let $N = N_{n, n_c}$. Now we work in a (sufficiently saturated) nonstandard model $M \succ (\omega, 0, 1, +, \cdot, <, \dots)$. Let $a \in M \setminus \omega$ and let

$$c^* : (n+1)^a \rightarrow n_c$$

be given. We can assume, by coding method, c^* lives in M . We want to find a c^* -monochromatic line in $(n+1)^a$. First fix an indiscernible sequence

$$d_0 < d_1 < \dots < d_i < \dots < a$$

over c^* . For numbers $s \leq t \leq u \leq v$ and x , let

$$C(s, t, u, v; x) = (n)_{t-s} \hat{\ } (x)_{u-t} \hat{\ } (0)_{v-u} = \underbrace{\langle n \dots n \rangle}_{t-s} \underbrace{\langle x \dots x \rangle}_{u-t} \underbrace{\langle 0 \dots 0 \rangle}_{v-u}.$$

Then, for $\bar{e} = e_0, \dots, e_{N-1} \in (n+1)^N$, $f(\bar{e})$ is the sequence

$$(n)_{d_0} \hat{\ } C(d_{0,1,2,3}; e_0) \hat{\ } C(d_{3,4,5,6}; e_1) \hat{\ } \dots \hat{\ } C(d_{3N-3, 3N-2, 3N-1, 3N}; e_{N-1}) \hat{\ } (0)_{a-d_{3N}},$$

of length a , where $d_{i,j,k,l}$ denotes the sequence d_i, d_j, d_k, d_l . For $\bar{e} \in (n+1)^N$, let \hat{e} be the sequence obtained from \bar{e} by replacing every term $e_i = n$ with 0. For example, if $\bar{e} = \langle n, n-1, n, 1, \dots \rangle$, then $\hat{e} = \langle 0, n-1, 0, 1, \dots \rangle$. \hat{e} belongs to n^N .

Claim A. $c^*(f(\bar{e})) = c^*(f(\hat{e}))$.

We assume $e_k = n$. Then the following equations are true.

$$\begin{aligned} c^*(f(\bar{e})) &= c^*(\dots \wedge C(d_{3k,3k+1,3k+2,3k+3}; n) \wedge \dots) \\ &= c^*(\dots \wedge C(d_{3k,3k+2,3k+2,3k+3}; n) \wedge \dots) \end{aligned} \quad (1)$$

$$= c^*(\dots \wedge C(d_{3k,3k+1,3k+1,3k+3}; n) \wedge \dots) \quad (2)$$

$$\begin{aligned} &= c^*(\dots \wedge C(d_{3k,3k+1,3k+2,3k+3}; 0) \wedge \dots) \\ &= c^*(f(\bar{e}')). \end{aligned} \quad (3)$$

The equality (1) holds because the two cells $C(d_{3k,3k+1,3k+2,3k+3}; n)$ and $C(d_{3k,3k+2,3k+2,3k+3}; n)$ are the same. The equality (2) holds because of the indiscernibility of \bar{d} over c^* . The equality (3) holds because the two cells $C(d_{3k,3k+1,3k+1,3k+3}; n)$ and $C(d_{3k,3k+1,3k+2,3k+3}; 0)$ are the same. (End of Proof of Claim)

Now we consider the coloring $c' : (n+1)^N \rightarrow n_c$ defined by $c'(\bar{e}) = c^*(f(\bar{e}))$. By our choice of N , if c' is restricted to the domain n^N , there is a word $w(x) \in (n \cup \{x\})^N \setminus n^N$ such that $\{w(i) : i \in n\}$ is c' -monochromatic. By Claim A and the choice of $w(x)$, $\{w(i) : i \in n+1\}$ is also monochromatic. Let $w^*(x)$ denote the word $f(w(x))$. It is a sequence in $((n+1) \cup \{x\})^{N^*} \setminus (n+1)^{N^*}$ and the following claim clearly holds.

Claim B. $\{w^*(i) : i \in n+1\}$ is c^* -monochromatic.

Now we have shown that the following statement holds in M :

$$\exists a, \forall c^* : (n+1)^a \rightarrow n_c, \exists w^*(x) \text{ s.t. } \{w^*(i) : i \in n+1\} \text{ is a singleton.}$$

Since ω is an elementary substructure of M , the same statement holds in ω . This provides the induction step of our proof. \square

References

- [1] Pierre Matet, Shelah's proof of the Hales-Jewett theorem revisited, European Journal of Combinatorics 28(2007) 1742-1745.
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